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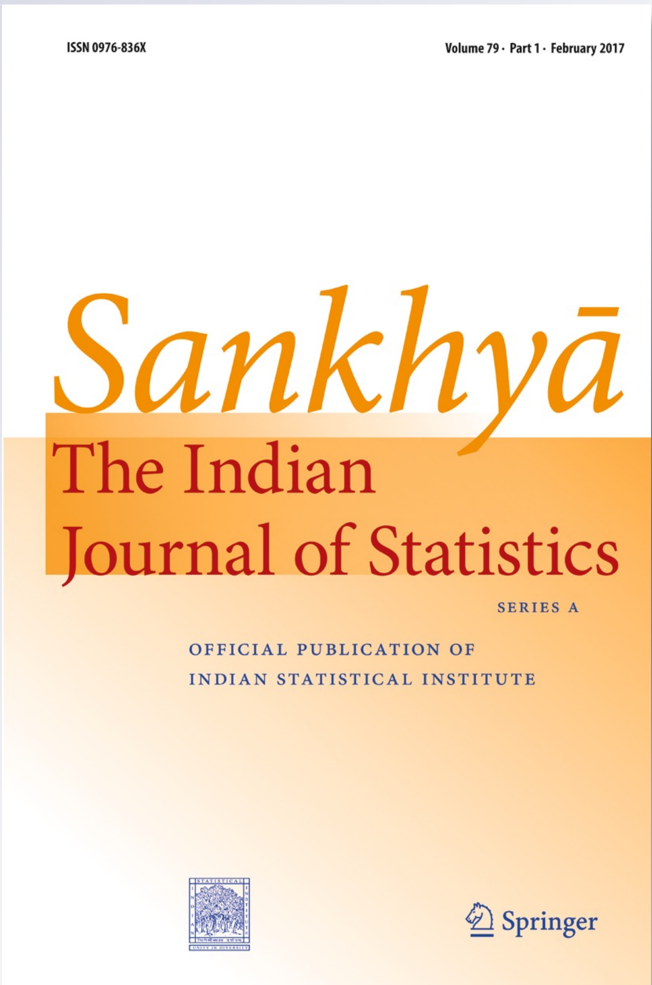

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A General Approach for Obtaining Wrapped Circular Distributions via Mixtures

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Abstract

We show that the operations of mixing and wrapping linear distributions around a unit circle commute, and can produce a wide variety of circular models. In particular, we show that many wrapped circular models studied in the literature can be obtained as scale mixtures of just the wrapped Gaussian and the wrapped exponential distributions, and inherit many properties from these two basic models. We also point out how this general approach can produce flexible asymmetric circular models, the need for which has been noted by many authors.

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1 Introduction

Circular stochastic models and their applications to various scientific disciplines is an important area of statistics, with various monographs devoted to this topic (see Mardia, 1972; Batschelet, 1981; Fisher, 1993; Mardia, and Jupp, 2000; Jammalamadaka and SenGupta, 2001; Pewsey et al., 2013). Circular distributions can be generated through various schemes, see e.g. Section 2.1.1. of Jammalamadaka and SenGupta (2001). Perhaps the most popular of these techniques is the idea of wrapping a linear distribution around the unit circle, giving rise to a multitude of wrapped circular distributions. Many wrapped versions of common probability distributions on the real line have been studied in the literature, including wrapped normal, Cauchy and stable distributions (see, e.g., Mardia and Jupp, 2000, pp. 50–52; Jammalamadaka and SenGupta, 2001, pp. 44–46; Gatto and Jammalamadaka, 2003; or Pewsey, 2008), wrapped exponential and Laplace distributions (see Jammalamadaka and Kozubowski, 2001, 2003, 2004), wrapped gamma distribution (see Coelho, 2011), wrapped t-distribution (see Kato and Shimizu, 2004; Pewsey et al., 2007), and wrapped lognormal, logistic, and Weibull models (see Rao et al., 2007; Sarma et al., 2011). We show that many of the wrapped circular models studied in literature are obtainable as *scale mixtures* of two basic models viz. the wrapped normal and the wrapped exponential distributions. We also point out that other asymmetric wrapped models (see Reed and Pewsey, 2009) can also be obtained by wrapping a *location-scale mixture* of normals.

Consider a family of probability distributions on the real line $\mathbb{R}$, indexed by $s \in \mathcal{S} \subset \mathbb{R}$, with probability density functions (PDF) $f(x|s)$, $x \in \mathbb{R}$. Let Y be a random variable whose PDF is a mixture, as follows:

$$f(y) = \int_{\mathcal{S}} f(y|s)dG(s), \quad y \in \mathbb{R}, \tag{1.1}$$

where G is a cumulative distribution function (CDF) on $\mathbb{R}$. This G can be a discrete distribution, leading to finite mixtures.

Using the notation f° to denote a circular density on $[0, 2\pi)$, the PDF of the wrapped distribution corresponding to the mixture distribution $f(y)$ is given by (see, e.g., Jammalamadaka and SenGupta, 2001, p. 31)

$$f^{\circ}(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi), \quad \theta \in [0, 2\pi), \tag{1.2}$$

while that corresponding to the component distribution $f(\cdot|s)$ is

$$f^{\circ}(\theta|s) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi|s), \quad \theta \in [0, 2\pi). \tag{1.3}$$

Crucial to our development is the Proposition 2.1, which states that the operations of mixing and wrapping commute in general, so that

$$f^{\circ}(\theta) = \int_{\mathcal{S}} f^{\circ}(\theta|s)dG(s), \quad \theta \in [0, 2\pi). \tag{1.4}$$

With an eye toward concrete applications, we now consider a random variable S with CDF G and PDF g. Let v and w denote appropriate location and scaling functions. When $\mathcal{S} = \mathbb{R}_+$ and X is an underlying "base" random variable with the PDF h, the PDF of the random variable

$$Y|s \stackrel{d}{=} v(s) + w(s)X, \tag{1.5}$$

is given by

$$f(y|s) = \frac{1}{w(s)} h\left(\frac{y - v(s)}{w(s)}\right), \quad y \in \mathbb{R}. \tag{1.6}$$

Then the relation (1.1) describes a rich class of what we call *location-scale mixtures* of X. Special cases of this include some well-known and important classes of distributions, such as

- *normal variance-mean mixtures*, where $v(s) = c + \mu s$, $w(s) = \sigma\sqrt{s}$, and X is standard normal (see, e.g., Barndorff-Nielsen et al., 1982),

- *Gaussian scale mixtures*, where $v(s) = 0$, $w(s) = \sigma\sqrt{s}$, and X is as above (see, e.g., Andrews and Mallows, 1974; West, 1984, 1987),

- *mixtures of exponential distributions*, where $v(s) = 0$, $w(s) = 1/s$, and X is an exponential variable (see, e.g., Jewell, 1982), and

- *location mixtures*, where $v(s) = s$ and $w(s) = 1$.

In this very general framework of location-scale mixtures (1.5), we use the idea of wrapping and apply it to mixture distributions given in (1.1), leading to rich classes of circular distributions. In general, as we demonstrate in Section 3, this scheme leads to *symmetric* as well as *asymmetric* distributions on the circle, including the wrapped skew Laplace model of Jammalamadaka and Kozubowski (2003, 2004) and the wrapped normal-Laplace circular model of Reed and Pewsey (2009). While asymmetric circular models can be produced by wrapping *asymmetric* linear distributions (see, e.g., Pewsey, 2000), this approach involving mixtures sheds more light on such models. A particular case of *scale* mixtures of normal and exponential distributions leads to the two families of *mixtures of wrapped normal* and *mixtures of wrapped exponential* circular distributions, discussed in detail in Section 4, both as applications and as a demonstration of the power of this idea, which can result in large classes of novel circular models.

The rest of the paper is organized as follows. Section 2 discusses main results regarding wrapping and mixtures. Section 3 considers wrapping of variance-mean mixtures of linear models, to produce a wide variety of symmetric and asymmetric circular models. Also included in this section is a discussion of how other asymmetric circular models can be obtained from mixtures of asymmetric linear models, like the skew-normal distribution. Section 4 demonstrates specifically how scale mixtures of normal distributions (the MWN family of Section 4.1) and scale mixtures of exponentials (the MWE family of Section 4.2) can be used to generate the wrapped models that have been discussed in the literature. Section 5 concludes with some final remarks.

2 Mixtures of Wrapped Distributions—the General Case

Let $\phi(t|s)$ denote the characteristic function (ChF) corresponding to the PDF $f(x|s)$. If Y is a random variable whose distribution is obtained by mixing these distributions with respect to G on $\mathcal{S}$, then the PDF of Y is given by (1.1) while its ChF is

$$\phi(t) = \mathbb{E}\exp(itY) = \int_{\mathcal{S}} \phi(t|s)dG(s), \ \ t \in \mathbb{R}. \tag{2.1}$$

By wrapping the distribution of Y around the unit circle, we obtain a circular counterpart of Y, denoted by Y°, whose ChF

$$\{\phi_p^\circ \ : p = 0, \pm 1, \pm 2 \ldots\} \tag{2.2}$$

is given by (see, e.g., Proposition 2.1 of Jammalamadaka and SenGupta, 2001, p. 31)

$$\phi_p^\circ = \phi(p) = \int_{\mathcal{S}} \phi(p|s)dG(s), \ \ p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}. \tag{2.3}$$

Then we have the following result, which states that the operations of wrapping and mixing commute.

PROPOSITION 2.1. *The circular distribution obtained by wrapping a mixture of linear distributions given by the PDF* (1.1) *around the unit circle, coincides with a mixture (with respect to the same mixing distribution G) of circular distributions with the PDFs* (1.3).

PROOF. From the definitions already given, we have

$$\begin{aligned} f^\circ(\theta) &= \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi) = \sum_{k=-\infty}^{\infty} \int_{\mathcal{S}} f(\theta + 2k\pi|s)dG(s) \\ &= \int_{\mathcal{S}} \left\{ \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi|s) \right\} dG(s) = \int_{\mathcal{S}} f^\circ(\theta|s)dG(s), \ \ \theta \in [0, 2\pi). \end{aligned}$$

The interchanging of the order of integration and summation is justified by the standard Fubini-Tonelli theorem as the integral is bounded and the functions under the integral are non-negative. This establishes the proposition and the result stated in (1.4).

REMARK 1. If G is a discrete distribution, the mixtures in the above setting are finite and such special cases of discrete mixtures of wrapped normal and wrapped exponential distributions are studied in Agiomyrgiannakis and Stylianou (2009) and Coelho (2011), respectively.

We now focus on the special case of continuous location-scale mixtures mentioned in the introduction. For simplicity we shall assume that G is absolutely continuous with respect to the Lebesgue measure, so that $dG(s) = g(s)ds$ (the discrete case is very similar). Since S is typically a scaling random variable, we will further assume that g is supported on $\mathbb{R}_+$. Using the specific form (1.6) in (1.1), the PDF of the random variable $Y = v(S) + w(S)X$, where X and S are independent with the PDFs h and g respectively, becomes

$$f(y) = \int_0^\infty \frac{1}{w(s)} h\left(\frac{y - v(s)}{w(s)}\right) g(s)ds, \;\; y \in \mathbb{R}. \tag{2.4}$$

Now, as a restatement of Proposition 2.1 for this particular case, we have

COROLLARY 2.1. *The wrapped version of the PDF* (2.4) *has the representation*

$$f^\circ(\theta) = \int_0^\infty f^\circ(\theta|s)g(s)ds, \;\; \theta \in [0, 2\pi), \tag{2.5}$$

where, for each $s \in \mathbb{R}_+$,

$$f^\circ(\theta|s) = \sum_{k=-\infty}^{\infty} \frac{1}{w(s)} h\left(\frac{\theta + 2k\pi - v(s)}{w(s)}\right), \;\; \theta \in [0, 2\pi). \tag{2.6}$$

It is easy to see that similar "mixture" representations apply to certain other characteristics of Y°, such as the ChF (2.2) of Y° , which admits the representation

$$\phi_p^\circ = \int_0^\infty \phi_{p|s}^\circ g(s)ds, \;\; p \in \mathbb{Z}, \tag{2.7}$$

where the quantity $\phi_{p|s}^\circ$ is the ChF of the wrapped version of $v(s) + w(s)X$, whose PDF is given in (1.6).

In the following two sections, we demonstrate how various wrapped circular models studied in the literature can be obtained in a unified way through mixtures, and how some of them can be extended to produce further circular models. In doing this, we provide streamlined alternate derivations for the wrapped PDFs via this scheme, instead of what we consider ad-hoc approaches used in each case.

3 Obtaining Symmetric and Asymmetric Circular Models via Mixing

Many authors have noted the need for flexible families of *asymmetric* circular models, see, e.g., Pewsey (2000) or Umbach and Jammalamadaka (2009). We now provide a general approach to build such models via wrapping and mixing.

3.1. Asymmetry via Location-Scale Mixing of Gaussian Models. Consider the mixture model (1.1) discussed in the introduction, where $f(x|s)$ is given by (1.6) with $v(s) = c + \mu s$ and $v(s) = \sigma\sqrt{s}$ for some $c, \mu \in \mathbb{R}$ and $\sigma > 0$. If the base random variable X with the PDF h is standard normal, we obtain the well-known class of variance-mean mixtures of normal distributions (see, e.g., Barndorff-Nielsen et al., 1982). Then, (1.1) represents the PDF of the random variable

$$Y \stackrel{d}{=} \sigma\sqrt{S}X + S\mu + c, \tag{3.1}$$

and S is a non-negative random variable with CDF G, independent of X. Since the parameter c controls only the location, for simplicity we set it equal to zero. In this case, the PDF of Y becomes

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi s}\sigma} e^{-\frac{(x-\mu s)^2}{2s\sigma^2}} dG(s), \ \ x \in \mathbb{R}. \tag{3.2}$$

One can define countless linear asymmetric random variables by making various choices for the distribution G, thus generalizing many standard symmetric Gaussian scale mixtures discussed later in Section 4.1. In turn, wrapping these skewed linear distributions around the unit circle, leads to new families of skewed circular distributions. Their PDFs can be expressed in the form

$$f^\circ(\theta) = \int_0^\infty f^\circ(\theta|s) dG(s), \ \ \theta \in [0, 2\pi), \tag{3.3}$$

where the quantity

$$f^\circ(\theta|s) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^\infty e^{-\frac{\sigma^2 p^2 s}{2}} \cos[p(\theta - \mu s)]\right), \ \ \theta \in [0, 2\pi), \tag{3.4}$$

is the PDF of the wrapped normal distribution, obtained by wrapping (the conditionally normal) variable $Y|s \stackrel{d}{=} \mu s + \sigma\sqrt{s}X$ around the unit circle.

Remark 2. Consider the sum $Y = Y_1 + Y_2$ of two independent random variables, where Y_1 is a variance-mean mixture of the form $Y_1 \stackrel{d}{=} \sigma_1\sqrt{S}X + S\mu_1 + c_1$ with standard normal X, while Y_2 is $N(\mu_2, \sigma^2)$. Such a sum Y is also a variance-mean mixture of the form (3.1), with $\sigma = \sigma_1$, $\mu = \mu_1$, $c = c_1 + \mu_2 - \mu_1(\sigma_2/\sigma_1)^2$, and $S = S_1 + (\sigma_2/\sigma_1)^2$. Consequently, one can create new wrapped circular distributions starting with Y instead of the wrapped version of Y_1, just by adding an independent Gaussian random variable. One such special case is the wrapped normal-Laplace (and normal-generalized Laplace) distribution considered in Reed and Pewsey (2009).

As an illustration, consider an *asymmetric Laplace* linear variable Y with the stochastic representation (3.1), where S is exponential with the PDF

$$g(s) = \frac{1}{2}e^{-s/2}, \quad s \in \mathbb{R}_+. \tag{3.5}$$

The PDF and the ChF of Y are given by (see Kotz et al., 2001, pp. 136-137)

$$f(x) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases} e^{-\frac{\kappa x}{\sigma}} & \text{for } x \geq 0 \\ e^{\frac{x}{\kappa\sigma}} & \text{for } x < 0 \end{cases} \tag{3.6}$$

and

$$\phi(t) = \frac{1}{1 - 2i\mu t + \sigma^2 t^2}, \quad t \in \mathbb{R}, \tag{3.7}$$

respectively, where

$$\kappa = \frac{\sqrt{\mu^2+\sigma^2} - \mu}{\sigma} \tag{3.8}$$

is a skewness parameter. Note that when $\mu = 0$ (so that $\kappa = 1$), Y has the classical symmetric Laplace distribution. Upon wrapping, the PDF of the *wrapped asymmetric Laplace* variable Y° takes the form (3.3), where $f^\circ(\theta|s)$ is as in (3.4) with g given by (3.5). Straightforward integration shows that in this case the PDF (3.3) becomes

$$f^\circ(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty}\alpha_p\cos(p\theta) + \beta_p\sin(p\theta)\right), \quad \theta \in [0, 2\pi), \tag{3.9}$$

where

$$\alpha_p = \frac{1+\sigma^2p^2}{(1+\sigma^2p^2)^2 + 4\mu^2p^2}, \quad \beta_p = \frac{2\mu p}{(1+\sigma^2p^2)^2 + 4\mu^2p^2} \tag{3.10}$$

are the trigonometric moments of the wrapped variable Y°. The latter were obtained directly from the wrapped Laplace ChF (2.2), where

$$\phi_p^\circ = \frac{1}{1 - 2i\mu p + \sigma^2p^2} = \alpha_p + i\beta_p, \quad p \in \mathbb{Z},$$

as in Jammalamadaka and Kozubowski (2003). Moreover, as shown in that paper, the density (3.9) admits the closed form

$$f^\circ(\theta) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\left\{\frac{e^{-\frac{\kappa\theta}{\sigma}}}{1 - e^{-\frac{2\pi\kappa}{\sigma}}} + \frac{e^{\frac{\theta}{\kappa\sigma}}}{e^{\frac{2\pi}{\kappa\sigma}} - 1}\right\}, \quad \theta \in [0, 2\pi), \tag{3.11}$$

which simplifies to (4.34) in the symmetric case with $\kappa = 1$. Other skew circular models can be derived along similar lines, although their PDFs might not admit explicit forms.

3.2. Asymmetry via Scale Mixing of Skew-Normal Laws. An alternative way of obtaining an asymmetric circular distribution is by starting with an asymmetric distribution on the real line, such as the skew-normal distribution of Azzalini (1985), and wrapping it, as in Pewsey (2000). As shown there, wrapping the skew normal distribution of the form

$$f(x|s) = \frac{2}{\sqrt{2\pi s}\sigma} e^{-\frac{(x-\mu)^2}{2s\sigma^2}} \Phi\left(\lambda \frac{x-\mu}{\sqrt{s}\sigma}\right), \tag{3.12}$$

where Φ is the standard normal CDF, one gets a circular model with the PDF

$$f^{\circ}(\theta|s) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} e^{-\frac{\sigma^2 p^2 s}{2}} \left\{\cos[p(\theta-\mu)] + \tau(\delta\sigma\sqrt{s}p)\sin[p(\theta-\mu)]\right\}\right), \tag{3.13}$$

where $\delta = \lambda/\sqrt{1+\lambda^2} \in (-1,1)$ and τ is an odd function on $\mathbb{R}$ defined as

$$\tau(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{u^2/2} du, \quad x > 0. \tag{3.14}$$

Mixtures of the form

$$Y \stackrel{d}{=} \sigma\sqrt{S}X + \mu, \tag{3.15}$$

where X is a standard *skew-normal* random variable given by (3.12) and S is a non-negative random scale factor with distribution G, independent of X, have been studied in the literature (see Branco and Dey, 2001; Kim and Genton, 2011). Such scale mixtures of skewed distributions can be wrapped to produce a wide variety of asymmetric circular models, which will have the mixture representation (3.3) with $f^{\circ}(\theta|s)$ given by (3.13).

3.3. Wrapping finite asymmetric mixtures. The linear asymmetric Laplace density (3.6) can be written as a mixture

$$f(x) = p_1 f_1(x) + p_2 f_2(x), \quad x \in \mathbb{R}, \tag{3.16}$$

where

$$f_1(x) = \frac{2}{\kappa} u\left(-\frac{x}{\kappa}\right), \quad x \in \mathbb{R}_- \text{ and } f_2(x) = 2\kappa u(\kappa x), \quad x \in \mathbb{R}_+ \tag{3.17}$$

are the PDFs of the variables $-\kappa|X|$ and $|X|/\kappa$, respectively, with X having a symmetric Laplace distribution with the PDF u (defined on the right-hand-side of (3.6) with $\kappa = 1$). The two weights in (3.16) are given by

$$p_1 = \frac{\kappa^2}{1+\kappa^2}, \quad p_2 = \frac{1}{1+\kappa^2}, \tag{3.18}$$

which ensures continuity of the PDF (3.16). This is an example of a general scheme (see, e.g., Fernandez and Steel, 1998), where two inverse scale factors, $\kappa > 0$ and $1/\kappa > 0$, are used to transform a symmetric distribution with the PDF u into a skewed one viz. (3.16)–(3.18). The Gaussian distribution, along with all its symmetric scale mixtures (including those discussed in Section 4.1), have skew analogs in this scheme, with a large body of literature devoted to this topic (see, e.g., Mudholkar and Hutson, 2000 for skew-normal, Fernandez and Steel, 1998 for skew-t, Kotz et al., 2001 for skew-Laplace, and Arellano-Valle et al., 2005 for skew-logistic). Such skew models go beyond Gaussian scale mixtures, and can be built from virtually any linear distribution.

In turn, the results presented here lead to a multitude of skew circular distributions with densities of the form

$$f^\circ(\theta) = p_1 f_1^\circ(\theta) + p_2 f_2^\circ(\theta), \;\; \theta \in [0, 2\pi), \tag{3.19}$$

where the weights are as in (3.18) with $\kappa \in \mathbb{R}_+$ and the f_1° and f_2° are the wrapped versions of the "one-sided" distributions with linear densities as in (3.17). In case u is a symmetric Laplace density, we obtain the wrapped (skew) Laplace distribution of Jammalamadaka and Kozubowski (2003) with an explicit density (3.11), while in general the PDFs (3.19) are only representable in terms of an infinite series. Still more general models of the form (3.19) can be obtained from (3.16) via wrapping, where the p_1 and p_2 are arbitrary, non-negative weights summing up to one while

$$f_1(x) = 2c_1 u\left(-c_1 x\right), \;\; x \in \mathbb{R}_- \;\text{ and }\; f_2(x) = 2c_2 u\left(c_2 x\right), \;\; x \in \mathbb{R}_+, \tag{3.20}$$

with arbitrary positive weights c_1, c_2 and u being the PDF of a symmetric distribution on $\mathbb{R}$.

4 Applications to Scale Mixtures of Gaussian and Exponential Distributions

While we presented very general methods for obtaining symmetric and asymmetric circular models via mixtures and wrapping in the earlier section, we like to get more specific, and consider simple *scale mixtures* of two basic distributions, namely the Gaussian and the exponential laws, and demonstrate how this unified approach helps streamline the derivation of various wrapped distributions considered in the literature, and alluded to in the introduction.

4.1. Scale Mixtures of Wrapped Normal Distributions. In this section, we consider the special case of scale mixtures of normal distributions (see, e.g., Andrews and Mallows, 1974; West, 1984, 1987), where in (3.1) we set $c = \mu = 0$ and choose X to be standard normal. Then the resulting variable

$$Y \stackrel{d}{=} \sigma\sqrt{S}X \tag{4.1}$$

is Gaussian with a "stochastic" variance given by $\sigma^2 S$. Observe that the PDF of Y is given by (3.2) with $\mu = 0$, leading to

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi s}\sigma} e^{-\frac{x^2}{2s\sigma^2}} g(s)ds, \quad x \in \mathbb{R}, \tag{4.2}$$

while the ChF of Y in (4.1) is given by

$$\phi(t) = \psi\left(\frac{t^2\sigma^2}{2}\right), \quad t \in \mathbb{R}, \tag{4.3}$$

where g is the PDF of S and

$$\psi(t) = \mathbb{E}e^{-tS} = \int_0^\infty e^{-ts} g(s)ds, \quad t \in \mathbb{R}_+, \tag{4.4}$$

is the corresponding Laplace Transform (LT) of S.

An application of the general results of Section 2 to this particular case sheds light on the nature of mixtures of wrapped normal distributions. In particular, major characteristics of the latter can be related to the LT (4.4) of the mixing variable S. First, observe that the ChF (2.2) of Y°, the wrapped counterpart of Y in (4.1), is real and given by

$$\phi_p^\circ = \psi\left(\frac{p^2\sigma^2}{2}\right), \quad p \in \mathbb{Z}, \tag{4.5}$$

with trigonometric moments

$$\alpha_p(Y^\circ) = \psi\left(\frac{p^2\sigma^2}{2}\right), \quad \beta_p(Y^\circ) = 0, \quad p \in \mathbb{Z}. \tag{4.6}$$

It follows that the mean direction is zero while the resultant length is

$$\rho = \psi\left(\frac{\sigma^2}{2}\right). \tag{4.7}$$

Further, by Corollary 2.1, the PDF of Y° admits the mixture representation (2.5), where $f^\circ(\cdot|s)$ is the PDF of the wrapped Gaussian distribution with mean zero and variance $\sigma^2 s$, given by (3.4) with $\mu = 0$. As shown by the following result, an alternative formula for the PDF of Y° through its Fourier coefficients (4.6) is available when

$$\sum_{p=1}^{\infty} \psi\left(\frac{p^2\sigma^2}{2}\right) < \infty. \tag{4.8}$$

PROPOSITION 4.1. *When the probability distribution of the random variable* (4.1), *given by the PDF* (4.2), *is wrapped around the unit circle, and the LT* (4.4) *of the variable S satisfies* (4.8), *then the PDF* (2.5) *of the resulting circular distribution admits the representation*

$$f^\circ(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} \psi\left(\frac{p^2\sigma^2}{2}\right)\cos(p\theta)\right), \quad \theta \in [0, 2\pi). \tag{4.9}$$

PROOF. The result follows by Lebesgue dominated convergence theorem upon substituting (3.4) with $\mu = 0$ into (2.5) and interchanging the order of integration and summation.

Before presenting special cases of some well-known wrapped distributions related to the above result, let us note the following special feature of scale mixtures of Gaussian distributions. If the variables Y_1 and Y_2 are independent, and each is a Gaussian scale mixture of the form $Y_i \stackrel{d}{=} \sigma_i\sqrt{S_i}X_i$ with standard normal X_i, independent of the random scale factor S_i, $i = 1, 2$, then their sum $Y \stackrel{d}{=} Y_1 + Y_2$ is also a Gaussian scale mixture of the form (4.1) with $\sigma = 1$ and $S \stackrel{d}{=} \sigma_1^2 S_1 + \sigma_2^2 S_2$. Consequently, one can generate multitude of new probability distributions of the form (4.1) from the existing Gaussian scale mixture models, including the ones presented below. In turn, their wrapped versions can be studied as mixtures of wrapped distributions along the lines discussed above. One such simple example is the normal-Laplace distribution studied in Reed (2007) and its wrapped (symmetric) version given in Reed and Pewsey (2009).

4.1.1. Wrapped t-Distribution. Consider the wrapped t-distribution (with location zero), obtained by wrapping a linear t-distribution with $\nu > 0$ degrees of freedom and scale parameter $\sigma > 0$ (see Pewsey et al., 2007). The latter is given by the PDF

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(1 + \frac{x^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R}. \tag{4.10}$$

It is well-known that a t-distributed random variable Y and its PDF (4.10) admit the scale mixture representations (4.1) and (4.2), respectively (see, e.g., Gneiting, 1997 and references therein). The "mixing" random variable S in (4.1) has an *inverse gamma* distribution with parameters $\alpha = \nu/2$ and $\beta = \nu/2$, given by the PDF

$$g(s) = \frac{\beta^\alpha}{\Gamma(\alpha)} s^{-\alpha-1} e^{-\beta/s}, \ \ s \in \mathbb{R}_+, \tag{4.11}$$

and the LT

$$\psi(t) = \frac{2}{\Gamma(\alpha)} \sqrt{t^\alpha \beta^\alpha} K_\alpha(2\sqrt{\beta t}), \ \ t \in \mathbb{R}_+, \tag{4.12}$$

where

$$K_\alpha(x) = \frac{1}{2} \int_0^\infty u^{\alpha-1} e^{-\frac{x}{2}\left(u+\frac{1}{u}\right)} du, \ \ x \in \mathbb{R}_+, \tag{4.13}$$

is the *modified Bessel function of the third kind* with index $\alpha \in \mathbb{R}$ (see, e.g., Gradshteyn and Ryzhik, 1994, p. 969). By setting $\alpha = \beta = \nu/2$ in (4.12), we obtain

$$\psi\left(\frac{\sigma^2 p^2}{2}\right) = \frac{K_{\nu/2}(p\sigma\sqrt{\nu})(p\sigma\sqrt{\nu})^{\nu/2}}{\Gamma(\nu/2) 2^{\nu/2-1}}, \ \ p \in \mathbb{Z}, \tag{4.14}$$

which, according to (4.5), coincides with the ChF of the wrapped t-distribution, with the resultant length given by (4.14) with $p = 1$. Note that formula (4.14) coincides with that given in Pewsey et al. (2007), obtained by a different method. Next, we show that the condition (4.8) holds with $\psi(p^2\sigma^2/2)$ given by (4.14).

Proposition 4.2. *For any $\nu, \sigma > 0$, let the quantity under the sum in (4.8) be defined as in (4.14). Then the condition (4.8) is fulfilled.*

Proof. In view of (4.14), we need to show that

$$\sum_{p=1}^\infty p^{\nu/2} K_{\nu/2}(2\gamma p) < \infty, \tag{4.15}$$

where $\gamma = \sigma\sqrt{\nu}/2 > 0$. By (4.13), we have

$$K_{\nu/2}(2\gamma p) = \int_0^\infty \psi(u)[h(u)]^p du, \tag{4.16}$$

where

$$\psi(u) = \frac{1}{2} u^{\frac{\nu}{2}-1} \ \text{ and } \ h(u) = e^{-\gamma\left(u+\frac{1}{u}\right)}.$$

Since both the above functions are continuous on $\mathbb{R}_+$ and the (positive) function h attains a unique maximum value at $u = 1$, the classical Laplace approximation (see, e.g., Wong, 1989, p. 56) leads to the asymptotic equivalence

$$\int_0^{\infty} \psi(u)[h(u)]^p du \sim \psi(1)[h(1)]^{p+1/2} \left[\frac{-2\pi}{ph''(1)}\right]^{1/2} \quad \text{as } p \to \infty. \tag{4.17}$$

When we note that $\psi(1) = 1/2$ and take into account that

$$h(1) = e^{-2\gamma} \text{ and } h''(1) = -2\gamma e^{-2\gamma},$$

in view of the relations (4.16) and (4.17), we conclude that

$$K_{\nu/2}(2\gamma p) \sim \frac{1}{2}\sqrt{\frac{\pi}{\gamma}} p^{-1/2} e^{-2\gamma p} \text{ as } p \to \infty. \tag{4.18}$$

Thus, the series in (4.15) is convergent, as desired.

By the above and Proposition 4.1, we conclude that the PDF

$$f^{\circ}(\theta) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma} \sum_{p=-\infty}^{\infty} \left(1 + \frac{(\theta + 2\pi p)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}, \ \theta \in [0, 2\pi), \tag{4.19}$$

of the wrapped t-distribution admits the representation

$$f^{\circ}(\theta) = \frac{1}{2\pi}\left(1 + \frac{(\sigma\sqrt{\nu})^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2-2}} \sum_{p=1}^{\infty} K_{\nu/2}(p\sigma\sqrt{\nu}) p^{\nu/2}\cos(p\theta)\right), \ \theta \in [0, 2\pi). \tag{4.20}$$

This is consistent with the fact that the wrapped t PDF (4.19) is a mixture of the form (2.5) of wrapped normal PDFs (3.4) with $\mu = 0$. We note that (4.20) generalizes the results of Kato and Shimizu (2004), containing a special case of wrapped t-distribution with *integer* degrees of freedom.

4.1.2. Wrapped Cauchy Distribution. In view of its many interesting properties and usefulness, we will briefly mention the wrapped Cauchy (WC) distribution although it is a special case of (4.10) when $\nu = 1$. Recall that the Cauchy distribution centered at zero has the PDF

$$f(x) = \frac{1}{\pi}\frac{\sigma^2}{\sigma^2 + x^2}, \ x \in \mathbb{R}. \tag{4.21}$$

The results for wrapped t-distribution with $\alpha = \beta = 1/2$ imply that the ChF (2.2) of the WC distribution is given by

$$\phi_p^{\circ} = \psi\left(\frac{\sigma^2 p^2}{2}\right) = e^{-\sigma|p|}, \ p \in \mathbb{Z}, \tag{4.22}$$

which also coincides with its trigonometric moment α_p (with β_p being zero). We conclude that the WC distribution is a mixture of wrapped normal distributions with PDFs (3.4) where $\mu = 0$, with its PDF admitting the representation (2.5) with

$$g(s) = \frac{1}{\sqrt{2\pi}} s^{-3/2} e^{-\frac{1}{2s}}, \ \ s \in \mathbb{R}_+. \tag{4.23}$$

Further, since (4.8) holds in the present case, the PDF of WC distribution can be written as

$$f^\circ(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} e^{-\sigma p}\cos(p\theta)\right), \ \ \theta \in [0, 2\pi), \tag{4.24}$$

which coincides with its well-known form. The PDF can also be expressed in the alternate form

$$f^\circ(\theta) = \frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos\theta}, \ \ \theta \in [0, 2\pi), \tag{4.25}$$

where $\rho = e^{-\sigma}$ (see, e.g., Jammalamadaka and SenGupta, 2001, p. 45).

4.1.3. Symmetric Wrapped Stable Distribution. The wrapped Cauchy distribution discussed in Section 4.1.2 is also a special case with $\alpha = 1$ of *symmetric wrapped stable* (SWS) distributions with general $\alpha \in (0, 2]$, obtained by wrapping linear symmetric stable laws around the unit circle (see, e.g., Gatto and Jammalamadaka, 2003; Jammalamadaka and SenGupta, 2001, p. 46; Mardia and Jupp, 2000, p. 52). The latter are most conveniently described in terms of their ChFs,

$$\phi(t) = e^{-\sigma^\alpha |t|^\alpha}, \ \ t \in \mathbb{R}, \tag{4.26}$$

where $\alpha \in (0, 2]$ is a tail parameter and $\sigma > 0$ is a scale parameter. For $\alpha = 2$ we obtain normal distribution with mean zero and variance $2\sigma^2$. When $\alpha < 2$, stable laws have infinite variance, and their mean is also infinite when $\alpha \leq 1$, which is the case for the Cauchy distribution. Upon wrapping, the ChF (2.2) of the SWS variable is given by (4.5) with

$$\psi(t) = e^{-(2t)^{\alpha/2}}, \ \ t \in \mathbb{R}_+. \tag{4.27}$$

This is the LT of $S = 2S_\beta$, where $\beta = \alpha/2$ and S_β is a standard *stable subordinator* with index $\beta \in (0, 1]$, given by the LT

$$\mathbb{E}e^{-tS_\beta} = e^{-t^\beta}, \ \ t \in \mathbb{R}_+. \tag{4.28}$$

Consequently, linear symmetric stable distributions are scale mixtures of normal distributions of the form (4.1) with the above S (cf., Gneiting, 1997 and references therein). We conclude that the SWS distribution is a mixture of wrapped normal distributions with PDFs (3.4) where $\mu = 0$, with its PDF admitting the representation (2.5) with g being the PDF corresponding to the LT (4.27). Further, since the condition (4.8) is fulfilled with ψ given by (4.27), the PDF of SWS distribution admits the representation

$$f^{\circ}(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty}\rho^{p^{\alpha}}\cos(p\theta)\right), \quad \theta \in [0, 2\pi), \tag{4.29}$$

with $\rho = \exp(-\sigma^{\alpha})$, which coincides with its well-known form (see, e.g., Jammalamadaka and SenGupta, 2001, p. 46).

4.1.4. Wrapped Laplace and Generalized Laplace Distributions. Consider a *generalized Laplace* linear random variable (see, e.g., Kotz et al., 2001, p. 180), given by the ChF

$$\phi(t) = \left(\frac{1}{1+\sigma^2 t^2}\right)^{\tau}, \quad t \in \mathbb{R}, \quad \sigma, \tau > 0. \tag{4.30}$$

The ChF (2.2) of the *wrapped generalized Laplace* (WGL) distribution is of the form (4.5) where

$$\psi(t) = \left(\frac{1}{1+2t}\right)^{\tau}, \quad t \in \mathbb{R}_+, \tag{4.31}$$

is the LT of gamma distribution with mean 2τ and the PDF

$$g(s) = \frac{1}{2^{\tau}\Gamma(\tau)} s^{\tau-1} e^{-s/2}, \quad s \in \mathbb{R}_+. \tag{4.32}$$

This reflects the well-known mixture representation (4.1) of the generalized Laplace distribution (see, e.g., Kotz et al., 2001, p. 183). We conclude that the WGL distribution is a mixture of wrapped normal distributions with PDFs (3.4) where $\mu = 0$, with its PDF admitting the representation (2.5) with g as in (4.32). In addition, it can be shown that for $\tau > 1/2$ the condition (4.8) is fulfilled, so that the PDF of WGL distribution can be written as

$$f^{\circ}(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty}\left(\frac{1}{1+\sigma^2 p^2}\right)^{\tau}\cos(p\theta)\right), \quad \theta \in [0, 2\pi). \tag{4.33}$$

In the special case $\tau = 1$ we obtain the *wrapped Laplace* (WL) distribution studied in Jammalamadaka and Kozubowski (2003, 2004), in which case the PDF takes on the explicit form

$$f^\circ(\theta) = \frac{1}{2\sigma}\left\{\frac{e^{-\theta/\sigma}}{1-e^{-2\pi/\sigma}} + \frac{e^{\theta/\sigma}}{e^{2\pi/\sigma}-1}\right\}, \quad \theta \in [0, 2\pi). \tag{4.34}$$

4.1.5. Wrapped Exponential Power Distribution. A generalization of WL distribution can be obtained by wrapping a symmetric *exponential power* linear random variable (see, e.g., Ayebo and Kozubowski, 2003), given by the PDF

$$f(x) = \frac{1}{2\sigma}\frac{\alpha}{\Gamma(1/\alpha)}e^{-|x/\sigma|^\alpha}, \quad x \in \mathbb{R}, \ \ 0 < \alpha < 2, \, \sigma > 0, \tag{4.35}$$

around the unit circle. Upon comparing the PDF (4.35) with the ChF (4.26), we see that exponential power and symmetric stable distributions are duals of each other. Consequently, the exponential power distribution with $\alpha \in (0, 2)$ is a scale mixture of normal distributions (4.1), and its density (4.35) admits representation (4.2), see, e.g., West (1987) or Gneiting (1997). The PDF of the "mixing" random variable S is of the form

$$g(s) = \sqrt{\frac{\pi}{2}}\frac{\alpha}{\Gamma(1/\alpha)}s^{-3/2}p_\alpha(1/s) \ \ s \in \mathbb{R}_+, \tag{4.36}$$

where p_α is the PDF of a positive stable random variable with the LT (4.27). Thus, the resulting *wrapped exponential power* (WEP) distribution is a mixture of wrapped normal distributions with PDFs (3.4) where $\mu = 0$, with its PDF admitting the representation (2.5) with g as in (4.36). Of course, this representation must coincide with the one obtained via (1.2), which in this case becomes

$$f^\circ(\theta) = \frac{1}{2\sigma}\frac{\alpha}{\Gamma(1/\alpha)}\sum_{k=-\infty}^{\infty} e^{-|\theta+2k\pi|^\alpha/\sigma^\alpha}, \quad \theta \in [0, 2\pi). \tag{4.37}$$

Note that in the special case $\alpha = 1$, the variable Y has the Laplace distribution, p_α in (4.36) is the Lévy PDF (4.23), and the PDF of S is simply the exponential PDF (3.5).

4.1.6. Wrapped Logistic Distribution. Our final example of mixtures of wrapped normal distributions is the *wrapped logistic* (WLG) distribution, which was treated briefly in Rao et al. (2007). Recall that the linear logistic distribution is given by the PDF (see, e.g., Johnson et al., 1995, Chapter 23)

$$f(x) = \frac{1}{\sigma}\frac{e^{-|x|/\sigma}}{\left(1+e^{-|x|/\sigma}\right)^2}, \quad x \in \mathbb{R}. \tag{4.38}$$

By taking into account the formula for the ChF corresponding to (4.38) (see, e.g., Ghosh et al., 2010), we conclude that the ChF (2.2) of the WLG circular model is of the form

$$\phi_p^\circ = \frac{\pi\sigma|p|}{\sinh(\pi\sigma|p|)}, \quad p \in \mathbb{Z}. \tag{4.39}$$

Further, it is well known that the logistic distribution (4.38) is a scale mixture of normal distributions of the form (4.1) (cf. Andrews and Mallows, 1974; Ghosh et al., 2010; or Stefanski, 1991), and its PDF (4.38) admits the representation (4.2) with

$$g(s) = \sum_{k=1}^{\infty}(-1)^{k-1}k^2 e^{-k^2 s/2}, \quad s \in \mathbb{R}_+. \tag{4.40}$$

It follows that the WLG distribution is a mixture of wrapped normal distributions with the PDFs $f^\circ(\theta|s)$ given by (3.4) where $\mu = 0$, with its PDF taking on the form (2.5). Now, in view of (4.39) and (4.3), we conclude that the LT of the variable S must be of the form

$$\psi(t) = \frac{\pi\sqrt{2t}}{\sinh(\pi\sqrt{2t})}, \quad t \in \mathbb{R}_+. \tag{4.41}$$

In turn, since the condition (4.8) holds with ψ given by (4.41), the PDF of WLG distribution can be written in the form

$$f^\circ(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty}\frac{\pi\sigma p}{\sinh(\pi\sigma p)}\cos(p\theta)\right), \quad \theta \in [0, 2\pi), \tag{4.42}$$

which was obtained directly from the trigonometric moments in Rao et al. (2007).

4.2. Scale Mixtures of Wrapped Exponential Distributions. Another class of circular distributions can be constructed by wrapping the distributions of exponential mixtures of the form

$$Y \stackrel{d}{=} \frac{X}{\lambda S}, \tag{4.43}$$

where $\lambda > 0$, the variable X is standard exponential, and S is a random variable on $\mathbb{R}_+$ with PDF g. The PDF and the ChF of Y are given by

$$f(y) = \int_0^\infty \lambda s e^{-\lambda s y} g(s) ds, \ \ y \in \mathbb{R}_+, \tag{4.44}$$

and

$$\phi(t) = \int_0^\infty \left(1 - \frac{it}{\lambda s}\right)^{-1} g(s) ds, \ \ t \in \mathbb{R}, \tag{4.45}$$

respectively. Note that, conditionally on $S = s$, we have (1.5) with $v(s) = 0$ and $w(s) = 1/\lambda s$, so that the wrapped version of Y in (4.43) has the wrapped exponential (WE) distribution given by the PDF (see Jammalamadaka and Kozubowski, 2001, 2003, 2004)

$$f^\circ(\theta|s) = \frac{\lambda s e^{-\lambda s \theta}}{1 - e^{-2\pi\lambda s}}, \ \ \theta \in [0, 2\pi). \tag{4.46}$$

Thus, according to Corollary 2.1, the PDF of the wrapped version of Y will be of the form

$$f^\circ(\theta) = \int_0^\infty \frac{\lambda s e^{-\lambda s \theta}}{1 - e^{-2\pi\lambda s}} g(s) ds. \tag{4.47}$$

Further, by (2.7), the trigonometric moments of Y° can be written as

$$\alpha_p(Y^\circ) = \int_0^\infty \frac{s^2}{s^2 + (p/\lambda)^2} g(s) ds, \ \ p \in \mathbb{Z}, \tag{4.48}$$

and

$$\beta_p(Y^\circ) = \frac{p}{\lambda} \int_0^\infty \frac{s}{s^2 + (p/\lambda)^2} g(s) ds, \ \ p \in \mathbb{Z}, \tag{4.49}$$

where we used expressions for the trigonometric moments of the WE distribution given in Jammalamadaka and Kozubowski (2003).

Before we present special cases of this scheme, let us note that all circular random variables Θ that one can define here via their PDFs as in (4.47), are *infinitely divisible*, i.e. for each integer $n \geq 1$, we have

$$\Theta \stackrel{d}{=} \Theta_1^{(n)} + \cdots + \Theta_n^{(n)} (\text{mod } 2\pi),$$

where the $\{\Theta_i^{(n)}, \ i \geq 1\}$ are some independent and identically distributed circular variables. This follows from the fact that all scale mixtures of exponential distributions are (linear) infinitely divisible (see, e.g., Steutel and van Harn, 2004, p. 334), and this property is retained upon wrapping (Mardia and Jupp, 2000, p. 48).

4.2.1. Wrapped Pareto Distribution. Suppose that the mixing variable S in (4.43) has a gamma distribution $G(1/\alpha, 1/\alpha)$, where $G(\alpha, \beta)$ denotes gamma distribution with the PDF

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \in \mathbb{R}_+. \tag{4.50}$$

Then, Y in (4.43) has Pareto Type II (Lomax) distribution with the PDF (see, e.g., Johnson et al., 1994, p. 575)

$$f(x) = \lambda(1 + \alpha\lambda x)^{-\frac{1}{\alpha}-1}, \quad x \geq 0. \tag{4.51}$$

This leads to a new family of wrapped Pareto (WP) distributions with the PDFs of the form:

$$f^\circ(\theta) = \frac{(1/\alpha)^{1/\alpha}}{\Gamma(1/\alpha)} \int_0^\infty \frac{\lambda s e^{-\lambda s\theta}}{1 - e^{-2\pi\lambda s}} s^{\frac{1}{\alpha}-1} e^{-s/\alpha} ds, \quad \theta \in [0, 2\pi), \tag{4.52}$$

and the ChF (2.2) given by

$$\phi_p^\circ = \frac{(1/\alpha)^{1/\alpha}}{\Gamma(1/\alpha)} \int_0^\infty \frac{1}{1 - \frac{ip}{\lambda s}} s^{\frac{1}{\alpha}-1} e^{-s/\alpha} ds, \quad p \in \mathbb{Z}. \tag{4.53}$$

In the special case $\alpha = 0$, the variable $S \sim G(1/\alpha, 1/\alpha)$ in (4.43) becomes a unit mass at 1, and the wrapped Pareto distribution reduces to the wrapped exponential.

A direct calculation based on the linear Pareto random variable Y with the PDF (4.51) leads to an alternative formula

$$f^\circ(\theta) = \lambda \left(\frac{1}{2\alpha\lambda\pi}\right)^{\frac{1}{\alpha}+1} \zeta\left(\frac{1}{\alpha} + 1, \frac{1 + \alpha\lambda\theta}{2\alpha\lambda\pi}\right), \quad \theta \in [0, 2\pi),$$

where

$$\zeta(r, q) = \sum_{k=0}^{\infty} \left(\frac{1}{k + q}\right)^r \tag{4.54}$$

is the generalized Riemann-Zeta function. Similar representations involving special functions were developed for wrapped t-distribution in Pewsey et al. (2007), and for wrapped gamma distribution in Coelho (2011), which we discuss below.

4.2.2. Wrapped Weibull Distribution. If the variable S in (4.43) is a stable subordinator with index $\beta \in (0,1]$ and the LT ψ given by the right-hand-side of (4.28), then Y in (4.43) has Weibull distribution with shape parameter $\beta \in (0,1]$, scale λ, and the PDF

$$f(x) = \beta\lambda^{\beta} x^{\beta-1} e^{-\lambda^{\beta} x^{\beta}}, \ \ x \in \mathbb{R}_{+}. \tag{4.55}$$

This leads to the family of *wrapped Weibull* (WW) distributions studied in Rao et al. (2007) and Sarma et al. (2011). According to the general results of this section, the PDF

$$f^{\circ}(\theta) = \beta\lambda^{\beta} \sum_{k=0}^{\infty} (\theta + 2k\pi)^{\beta-1} e^{-\lambda^{\beta}(\theta+2k\pi)^{\beta}}, \ \ \theta \in [0, 2\pi),$$

of the WW distribution, which appears in Rao et al. (2007), admits the representation (4.47) with g being the PDF of the stable subordinator with index β. In turn, the ChF (2.2) of WW distribution, whose derivation was one of the primary goals of Sarma et al. (2011), admits the representation

$$\phi_p^{\circ} = \int_0^{\infty} \frac{\lambda s}{\lambda s - ip} g(s) ds, \ \ p \in \mathbb{Z}. \tag{4.56}$$

The trigonometric moments of the WW distribution follow the representations (4.48) and (4.49) with the same g as well. Of course, in the special case $\beta = 1$, the WW distribution reduces to the wrapped exponential.

4.2.3. Wrapped Gamma Distribution. Let Y have a gamma $G(\alpha, \lambda)$ distribution with shape parameter $\alpha > 0$ and scale $\lambda > 0$. It is well-known that such a variable is a scale mixture of exponentials, but only if $\alpha \in (0,1)$ (see Gleser, 1987). This fact can most easily be observed by first denoting $Y = T_\alpha/\lambda$, where T_α has a standard gamma distribution $G(\alpha, 1)$, and then by expressing T_α as a product of two independent quantities (see, e.g., Johnson et al., 1994, p. 350),

$$T_\alpha = \left(\frac{T_\alpha}{T_\alpha + T_{1-\alpha}}\right)(T_\alpha + T_{1-\alpha}). \tag{4.57}$$

Here, $T_{1-\alpha}$ is independent of T_α and has a standard gamma distribution with shape parameter $1-\alpha$,

$$V = \frac{T_\alpha}{T_\alpha + T_{1-\alpha}} \tag{4.58}$$

has a beta distribution $B(\alpha, 1-\alpha)$ with the PDF

$$f_V(v) = \frac{v^{\alpha-1}(1-v)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}, \quad v \in (0,1), \tag{4.59}$$

and $X = T_\alpha + T_{1-\alpha}$ has the standard exponential distribution. Thus, for $\alpha \in (0,1)$, we have the representation (4.43) with $S \stackrel{d}{=} 1/V$, so that the PDF of S is given by

$$g(s) = \frac{s^{-1}(s-1)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}, \quad s \in (1,\infty), \tag{4.60}$$

as observed by Gleser (1987).[1] Consequently, according to (4.47), the PDF of the *wrapped gamma* (WG) distribution with $\alpha \in (0,1)$ and $\lambda > 0$ is of the form

$$f^\circ(\theta) = \frac{\lambda}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_1^\infty \frac{e^{-\lambda s\theta}}{1-e^{-2\pi\lambda s}} \left(\frac{1}{s-1}\right)^\alpha ds, \quad \theta \in [0, 2\pi), \tag{4.61}$$

with the corresponding ChF (2.2) given by

$$\phi_p^\circ = \frac{\lambda}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_1^\infty \frac{1}{\lambda s - ip} \left(\frac{1}{s-1}\right)^\alpha ds, \quad p \in \mathbb{Z}. \tag{4.62}$$

Further, according to (4.48) and (4.49), the trigonometric moments of this distribution are given by

$$\alpha_p(Y^\circ) = \frac{\lambda^2}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_1^\infty \frac{s}{s^2\lambda^2+p^2} \left(\frac{1}{s-1}\right)^\alpha ds, \quad p \in \mathbb{Z}, \tag{4.63}$$

and

$$\beta_p(Y^\circ) = \frac{\lambda p}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_1^\infty \frac{1}{s^2\lambda^2+p^2} \left(\frac{1}{s-1}\right)^\alpha ds, \quad p \in \mathbb{Z}, \tag{4.64}$$

respectively. Upon computing the above integrals, facilitated by integration formula 12 on p. 345 of Gradshteyn and Ryzhik (1994), we arrive at the closed-form expressions

$$\alpha_p(Y^\circ) = \lambda^\alpha(\lambda^2+p^2)^{-\alpha/2}\cos[\alpha\arctan(p/\lambda)], \quad p \in \mathbb{Z},$$
$$\beta_p(Y^\circ) = \lambda^\alpha(\lambda^2+p^2)^{-\alpha/2}\sin[\alpha\arctan(p/\lambda)], \quad p \in \mathbb{Z},$$

which were obtained directly from the WG ChF

$$\phi_p^\circ = \left(\frac{\lambda}{\lambda - ip}\right)^\alpha = \lambda^\alpha(\lambda^2+p^2)^{-\alpha/2}e^{i\alpha\arctan(p/\lambda)}, \quad p \in \mathbb{Z}, \tag{4.65}$$

[1]There seems to be a misprint in formula (3) of Gleser (1987).

in Coelho (2011). The latter work also contains a straightforward derivation of the WG PDF directly from the linear gamma PDF, leading to

$$f^{\circ}(\theta)=\sum_{k=0}^{\infty}f(\theta+2k\pi)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}e^{-\lambda\theta}(2\pi)^{\alpha-1}\Phi\left(e^{-2\lambda\pi},1-\alpha,\frac{\theta}{2\pi}\right),$$

where

$$\Phi(z,r,q)=\sum_{k=0}^{\infty}\frac{z^k}{(k+q)^r} \tag{4.66}$$

is the Lerch's transcendental special function.

5. Summary and Concluding Remarks

In this paper we raise the general question, that if we start with a mixture of component distributions on the real line and obtain a circular distribution by wrapping such a mixture around the circle, does this correspond to the mixture of component-wise wrapped distributions? We answer this question in the affirmative and show that in general these two operations commute, and can produce a multitude of symmetric as well as asymmetric circular models. In particular, wrapping general location-scale mixtures of Gaussian distributions generates many flexible asymmetric circular models, the need for which has been noted by many authors.

We note that scale mixtures of normal and exponential distributions play a central role, and discuss these specific families viz. (i) *mixtures of wrapped normal* (MWN) and (ii) *mixtures of wrapped exponential* (MWE) circular distributions, in detail. We show that the MWN and MWE distributions can serve as the basic building blocks in constructing many wrapped families that have been introduced in the literature over the years. To demonstrate the power of this idea, we show that the wrapped versions of Cauchy, Laplace, symmetric stable, logistic, and t-distribution are special cases of MWN, while the wrapped versions of gamma, Weibull, and Pareto distributions are special cases of MWE. Clearly, not every circular distribution can be generated by wrapping or considered as a mixture of wrapped normal or exponential distributions, except in a trivial sense.

The overall theme of the paper is that there is a unified treatment to creating a multitude of circular models via the method of wrapping mixtures. The results presented here can be used to obtain useful asymmetric circular models. Potential extensions of the approach to *multivariate* circular or cylindrical wrapped distributions, or directional distributions on other manifolds such as the torus, is an interesting topic for further research.

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